DISCRETE APPLIED MATHEMATICS

ELSEVIER Discrete Applied Mathematics 93 (1999) 141–148

Bipartite embeddings of trees in the plane ☆

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Received 30 April 1997; revised 28 July 1998; accepted 23 November 1998

Abstract

In this paper we consider the following embedding problem. A point set P in the plane in general position is partitioned into two disjoint sets R and B, and we are asked to embed a tree T in P without crossings and with the additional property that all the edges connect a point in R to another point in B. We study several problems related to such *bipartite embeddings*.

1. Introduction

Given a tree T on n vertices and a set P of n points in the plane in general position, it is known that T can be straight line embedded in P without crossings (see Lemma 14.7 in [8] for a stronger result). The problem becomes more difficult if T is rooted and we want to root it at any particular point of P. The problem in this form was posed by Perles and partially solved by Pach and Töröcsick [9]. A complete solution was found by Ikebe et al. [7]. A related result by A. Tamura and Y. Tamura [10] is that, given a point set $P=\{p_1,\dots,p_n\}$ and a sequence $d=(d_1,\dots,d_n)$ of positive integers with $\sum d_i=2n-2$, there exists an embedding of some tree in P such that the degree of p_i is equal to d_i. Optimal algorithms for solving the above problems have been found by Bose, McAllister and Snoeyink [3].

In this paper we consider the following embedding problem. A point set P in the plane in general position (no three points collinear) is partitioned into two disjoint sets R and B (the *red* and the *blue* points), and we are asked to embed a tree T in P

☆ Research supported partially by Proyecto DGES-MEC-PB96-0005-C02-02.

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PII: S0166-218X(99)00042-6

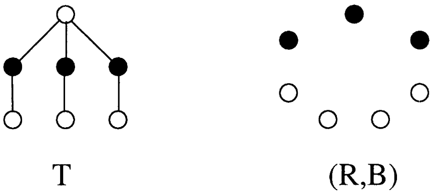

Fig. 1. T does not admit a bipartite embedding with respect to (R,B).

without crossings and with the additional property that all the edges connect a point in R to another point in B. We call such an embedding a *bipartite embedding* of T with respect to the bipartition (R,B).

A tree has a unique bipartition, and an obvious necessary condition for the existence of a bipartite embedding of T in P is that the cardinalities of the bipartitions of T and of P match correctly. However, simple examples show that this is not always sufficient (Fig. 1).

On the other hand, given a bipartition (R,B), is it always possible to find a bipartite embedding of *some* tree with respect to (R,B): take any red point and join it to all the blue points, then connect the remaining red points to suitable blue points without creating crossings. However, this simple solution produces trees with very large maximum degree.

Our approach here is to consider several natural embedding problems and investigate for which bipartitions they can be solved. In what follows the bipartition (R,B) is given as input, and $r=|R|$, $b=|B|$.

Problem 1 (Bounded degree embeddings). Given a bipartition (R,B) with $r \leqslant b$, find a bipartite embedding of a tree with respect to (R,B) having maximum degree $\Delta = \mathrm{O}(b/r)$.

Problem 2 (Fixed degree embeddings). Given a bipartition (R,B), $R=\{p_1,\dots,p_r\}$, and a sequence of positive integers $(d_1,\dots,d_r)$ with $\sum d_i = r+b-1$, find a bipartite embedding of a tree with respect to (R,B) such that the degree of p_i is d_i.

Problem 3 (Embedding of a spanning path). Find sufficient conditions for a bipartition (R,B) with $|r-b| \leqslant 1$ to admit a bipartite embedding of a spanning path.

Some comments are in order. The bound $\mathrm{O}(b/r)$ in Problem 1 is clearly best possible. In the general case we are able to establish it with a logarithmic overload. In Problem 2 one cannot fix the degree sequences of both R and B, as simple examples demonstrate. Finally, the condition $|r-b| \leqslant 1$ in Problem 3 is clearly necessary, but not sufficient as the example in Fig. 2 shows: the first edge of the path has to be an edge of the convex hull and the fact that the cardinalities of consecutive red and blue chains differ always in at least two units, prevents the path from spanning all the vertices.

The paper is organized as follows. Section 2 contains the results for points in general position, while in Section 3 we restrict our attention to particular configurations of points. A preliminary version of this paper appeared in [1].

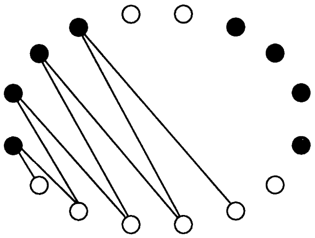

Fig. 2. The bipartite path cannot be completed to a spanning path.

2. Points in general position

In this section we present two results that hold for any bipartition in general position. The first one provides a partial answer to Problem 1. But first we need a lemma.

Lemma 2.1. *Given two disjoint bipartite embeddings* T_1 *and* T_2 *separated by a straight line, there exist an edge e between* T_1 *and* T_2 *such that* $T_1 \cup T_2 \cup \{e\}$ *is also a bipartite embedding.*

Proof. Let p be the point in T_1 closest to the separating line. If we consider T_2 as a collection of segments, then p has to see completely one of the segments in T_1, hence it can be connected to one of the two extremes, namely the one with opposite colour. □

Theorem 2.2. *Given a bipartition* (R,B) *in the plane with* $r \leqslant b$, *one can find a bipartite embedding of a tree with respect to* (R,B) *such that the maximum degree* Δ *is* $\mathrm{O}(b/r + \log r)$.

Proof. We use the ham-sandwich theorem [4], namely that the sets R and B can be simultaneously bisected by a straight line. Assume for simplicity that $r = 2^k$ and that $b = \alpha r = \alpha 2^k$. Applying repeatedly the ham-sandwich theorem we arrive, after k steps, to a partition of the plane into convex polygonal regions, all of them containing exactly one red point and α blue points. Join every red point to the corresponding α blue points to obtain a collection of r disjoint copies of a bipartite embedding of a star $K_{1,\alpha}$.

Using Lemma 2.1, merge these r partial trees into a single tree, in the opposite order as they have been produced, preserving the bipartite character of the embedding. Finally perform k merging steps, every time reducing by half the number of trees. It is clear that at every step the maximum degree increases at most by one. Since initially $\Delta = \alpha$, at the end $\Delta \leqslant \alpha + k = b/r + \log_2 r$. The cases where r is not a power of two or b/r is not an integer are treated similarly. □

In the next section it will be shown that the optimal bound $\mathrm{O}(b/r)$ can be achieved for several particular configurations. Our next result is that Problem 2 always admits a solution.

Theorem 2.3. *Given a bipartition* (R,B) *in the plane, with* $R=\{p_1,\dots,p_r\}$ *and a sequence* $(d_1,\dots,d_r)$ *of positive integers with* $\sum d_i=r+b-1$, *there exists a bipartite embedding of a tree with respect to* (R,B) *such that the degree of* p_i *is equal to* d_i.

Proof. By induction on $r+b$. Assume without loss of generality that $d_1=\max d_i$, and let l be an oriented line through p_1 not containing any other point from R or B. Let H^+ and H^- be the right and left open halfspaces in which the line l divides the plane. Let $r^+=|R\cap H^+|$ and $b^+=|B\cap H^+|$, and let also $r^-=|R\cap H^-|$ and $b^-=|B\cap H^-|$. Finally, define two functions (that depend on l) f^+ and f^- as follows:

$$f^+ = \sum_{p_i\in H^+} d_i - r^+ - b^+;$$

$$f^- = \sum_{p_i\in H^-} d_i - r^- - b^-,$$

and observe that $f^+ + f^- = -d_1$.

We claim that there exists some position of l in which $-d_1 < f^+ < 0$. To prove the claim assume that initially $f^+ \geqslant 0$ and consider the changes in f^+ as l turns around p_1.

If a red point p_i enters H^+, then f^+ increases by d_i-1;
If a red point p_i exits H^+, then f^+ decreases by d_i-1;
If a blue point enters H^+, then f^+ decreases by 1;
If a blue point exits H^+, then f^+ increases by 1.

In any case the change in absolute value is at most $d_i-1\leqslant d_1-1$. Since after a turn of 180 degrees the values of f^+ and f^- are interchanged, and $f^+=-d_1-f^-$, it follows that $f^+\leqslant -d_1$. All this implies that for some intermediate value we have $-d_1<f^+<0$. If we assume instead that initially $f^+\leqslant -d_1$ we proceed in the same way, and the claim is proved.

Now by induction we can find a bipartite embedding of a tree with respect to the bipartition $((R\cap H^+)\cup\{p_1\}, B\cap H^+)$ in which p_1 has degree $-f^+$ and p_i has degree d_i for $p_i\in R\cap H^+$. Similarly we get a tree on H^- in which p_1 has degree $-f^-$, and the union of the two trees does the job. □

3. Points in restricted positions

We have already mentioned that a bipartition does not always admit a spanning path, and that Theorem 2.2 does not give the best possible bound for the maximum degree. It is then natural to restrict the geometry of the problem in order to obtain positive results. We consider three such restrictions, or particular positions: when R and B are separated by a straight line; when $R\cup B$ is a set in convex position; and finally when the vertices of R define a convex polygon containing all the vertices in B.

3.1. Linearly separable partitions

We say that a bipartition (R,B) is *linearly separable* if there exists a straight line separating R and B. Equivalently, if the convex hulls of R and B are disjoint. The next result can be deduced from the results in [6]. However, we present a short proof of it.

Theorem 3.1. *Every linearly separable bipartition* $P = R \cup B$ *with* $|r-b| \leqslant 1$ *admits a bipartite embedding of a spanning path.*

Proof. Assume without loss of generality that R and B are separated by a horizontal line, and let $p_1 \in R$ and $q_1 \in B$ be such that p_1q_1 is the left red/blue edge of the convex hull of P. We say that p_1q_1 is the left *bridge* of P. The initial point of the spanning path will be p_1 if $b = r-1$, q_1 if $b = r+1$, and either p_1 or q_1 if $b = r$. Assume we start at p_1 and set $C = \{p_1\}$ (C is an ordered list that corresponds to the spanning path as it is constructed). It is essential not to add both p_1 and q_1, because doing so one cannot guarantee that the path can be continued. At every step compute the left bridge pq of $P \backslash C$, and add p to C if the last point in C is in B, or add q if the last point in C is in R. In this way we get a bipartite embedding of a path that has no crossings because C is disjoint from the convex hull of $P \backslash C$ and hence from all edges added to it during the algorithm. □

The technique in the previous proof can also be used to obtain the following result.

Theorem 3.2. *Every linearly separable bipartition* $P = R \cup B$ *with* $r \leqslant b$ *admits a bipartite embedding of a tree* T *with* $\Delta(T) \leqslant 1 + \lceil (b-1)/r \rceil$.

Proof. Let $\bar{d} = 1 + \lceil (b-1)/r \rceil$. The idea is to find a bipartite embedding of a tree with respect to (R,B) in which the degrees in R are equal to $\bar{d}$ or to $\bar{d}-1$ and, at the same time, be able to bound the degrees in B.

First we can suppose that $\bar{d} > 2$, that is $b-1 > r$. Otherwise, since we are assuming $b \geqslant r$, Theorem 3.1 implies the existence of a bipartite embedding with $\Delta = 2$ and the theorem holds. Now find a sequence $(d_1, \ldots, d_r)$ of positive integers such that $\sum d_i = r+b-1$ and $d_i = \bar{d}$ or $d_i = \bar{d}-1$. We could use Theorem 2.3 in order to find a bipartite embedding realizing this degree sequence, but then we would not be able to control the degrees in B.

Instead we proceed as follows. Find a point p_1 in R such that there is a line separating p_1 and d_1 points $q_1, \ldots, q_{d_1}$ in B from the remaining points, where the q_j are sorted in polar order with respect to p_1. This is always possible taking left bridges as in the proof of the previous theorem. Join p_1 to $q_1, \ldots, q_{d_1}$, remove all these points except q_{d_1} from the bipartition and find a new point p_2 in R that can be separated together with to d_2 points in B. If we repeat this process, at the end we get a single point p_r in R and d_r points in B (see Fig. 3 for an illustration). The fact that $d_i > 1$ for every i, implies that the degrees of the points in B are equal to one or two. □

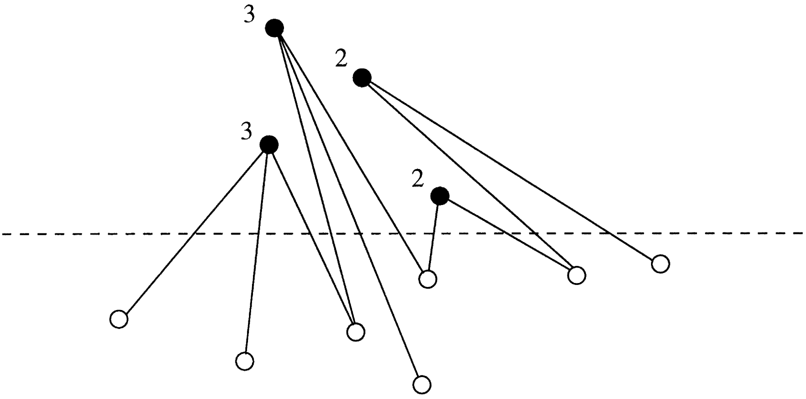

Fig. 3. Realizing the degree sequence $(3,3,2,2)$.

3.2. Sets in convex position

We say that a set of points is in convex position if it is the vertex set of a convex polygon. Problem 3 in this case was solved by Akiyama and Urrutia [2]. They found an $O(n^2)$ algorithm that determines if there is a bipartite spanning path, and finds such a path if it exists. Next we prove that Problem 1 is solvable.

Theorem 3.3. *Let $R \cup B$ be a set in convex position with $r \leqslant b$. Then the bipartition (R,B) admits a bipartite embedding of a tree T with $\Delta(T) \leqslant \lceil b/r \rceil + 2$.*

Proof. Assume for simplicity that $r=2^k$ and that $b=\alpha 2^k$. Using ham-sandwich cuts as in the proof of Theorem 2.2 we can obtain r disjoint red/blue copies of a star $K_{1,\alpha}$. The key point in this case is that we can control the degrees in the merging step.

Set initially T equal to any of the r stars. For every edge e of the convex hull of T that is not an edge of the convex hull of $R \cup B$, consider the trees $T_1,\dots,T_j$ that are visible from e and lie on the halfspace determined by e not containing T, ordered clockwise (this makes sense because the set is in convex position).

Next select one of the vertices of e and construct a bipartite polygonal chain connecting T and the trees $T_1,\dots,T_j$ (see Fig. 4). Because the set is in convex position, we can construct this chain in such a way that the degree of any vertex increases by at most 2. Set T equal to the tree obtained with the above construction and iterate the process until T is a spanning tree.

To see that the condition on the degree is satisfied, observe that the new edges of $CH(T)$ that are not in $CH(R \cup B)$ are determined by two vertices that belong to one of the trees $T_1,\dots,T_j$. Since only one vertex of each tree is used when constructing the polygonal chain, we can guarantee that we always have a free vertex to iterate the process. Therefore, points in R have degree at most $\alpha+2=b/r+2$, and points in B have degree at most 3. Observe finally that $3 \leqslant \lceil b/r \rceil + 2$. $\square$

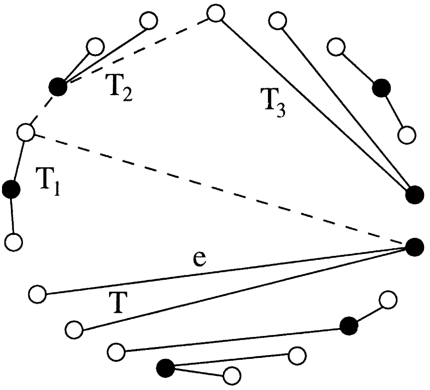

Fig. 4. Updating T in the proof of Theorem 4.3.

3.3. Bipartitions in which $R = CH(R \cup B)$

The situation can be described in this way: the points of R are the vertices of a convex polygon containing the points of B. We need the following result by García and Tejel [5].

Lemma 3.4. *Let P be a set of points in general position, and assume that $CH(P) = \{p_1, \ldots, p_r\}$ and that there are n interior points. Let $n = n_1 + \cdots + n_r$, where the n_i are positive integers. Then the convex hull of P can be partitioned into r convex polygons $Q_1, \ldots, Q_r$ such that Q_i contains n_i points and $p_i p_{i+1}$ is an edge of Q_i.*

Proof (*Sketch of the proof by García and Tejel*). By induction on r. If $r = 3$, by continuity arguments and due to the generic position of points, it is easy to prove that there exists a point $q \notin P$ inside the triangle $p_1 p_2 p_3$ such that the triangles $p_1 p_2 q$, $p_2 p_3 q$ and $p_3 p_1 q$ contain n_1, n_2 and n_3 points, respectively.

The induction step begins by considering an arbitrary diagonal, say $p_1 p_j$, with $1 < j < k$. Assume without loss of generality that the polygon $p_1 p_2 \cdots p_j$ contains $n_1 + \cdots + n_{j-1}$, or more points. Apply the induction hypothesis to this polygon and let $p_1 q_1 \cdots q_i p_j$ be the polygon obtained in the previous decomposition corresponding to the edge $p_1 p_j$. If this polygon contains n_j or more points, connecting p_{j+1} with a point q' in the polygonal chain $p_1 q_1 \cdots q_i p_j$, the polygon splits into two parts satisfying the induction hypothesis. Otherwise it is the polygon $p_1 p_{j+1} \cdots p_r$ which satisfies such condition. □

Using the above lemma we can solve Problems 1 and 3.

Theorem 3.5. *Let (R, B) be a bipartition in which $R = CH(R \cup B)$ and with $r \leqslant b$. Then it admits a bipartite embedding of a tree T with $\Delta(T) \leqslant 1 + \lceil (b-1)/r \rceil$.*

Proof. If $b = r\alpha + 1$ we can use Lemma 3.4 to decompose the convex hull of $R = \{p_1, \ldots, p_r\}$ into r convex polygons $Q_1, \ldots, Q_r$, such that Q_j contains α points of B for $j \neq 1$, Q_1 contains $\alpha + 1$ points of B, and $p_i p_{i+1}$ is an edge of Q_i (index arithmetic is modulo r).

Let s_j be the point in $Q_j \cap B$ closest to p_j, and join every red point p_j to the corresponding α blue points in Q_{j-1}, with the exception of s_1. Next merge these r partial trees by connecting s_j to p_j, for every $j = 1, \ldots, r$. In this way we obtain a tree T with $\Delta(T) = 1 + \alpha$. Finally, if $(b-1)/r$ is not an integer then we obtain a tree with $\Delta(T) \leqslant 1 + \lceil (b-1)/r \rceil$. □

Corollary 3.6. *Let* (R, B) *be a bipartition with* $R = CH(R \cup B)$ *and* $|r - b| \leqslant 1$. *Then it admits a bipartite embedding of a spanning path.*

Proof. If we assume $r \leqslant b$, then $b = r$ or $b = r + 1$ and the above theorem gives a tree with maximum degree 2, i.e. a spanning path. □

Acknowledgements

We thank Jorge Urrutia and Sue Withesides for their comments and for pointing out relevant references.

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